MA113. For any positive number t, $\lfloor t \rfloor$ denotes the integer part of t and $\{t\}$ denotes the "decimal" part of t. If $x + \{y\} = 7.32$ and $y + \lfloor x \rfloor = 8.74$, then determine $\{x\}$.

Originally from 2000 ESSO-CMS Math Camp, Problem Set 3, problem 3.

We received 8 correct solutions for this problem. We present the solution by the Missouri State University Problem Solving Group.

There are integer n and m such that $n \leq x < n+1$ and $m \leq y < m+1$. Then $\lfloor x \rfloor = n$ and $\{y\} = y - m$. The two given equations can be written as

$$\begin{aligned} x + y - m &= 7.32, \\ y + n &= 8.74. \end{aligned}$$

Subtracting, we get

$$x - n - m = -1.42, \quad \text{or} \quad x + 1.42 = n + m.$$

Since $n + m$ is an integer, we must have $\{x\} = 0.58$.

Remark. Since $0 \leq \{y\} < 1$, we have $6.32 \leq x < 7.32$, hence $x = 6.58$ and $y = 2.74$.

MA114. Let p, q, and r be positive constants. Prove that at least one of the following equations has real roots.

$$\begin{aligned} px^2 + 2qx + r &= 0 \\ rx^2 + 2px + q &= 0 \\ qx^2 + 2rx + p &= 0 \end{aligned}$$

Originally from 1983 Descartes Contest, problem 6b.

We received 9 correct solutions and one incorrect solution. Both solutions below assume only that p, q, r are real.

Solution 1, by Vikas Kumar Meena and Dominique Mouchet (done independently).

Adding the discriminants of the three equations, we get

$$4(q^2 - pr) + 4(p^2 - rq) + 4(r^2 - qp) = 2[(p-q)^2 + (q-r)^2 + (r-p)^2] \geq 0,$$

whence at least one of the discriminants is nonnegative. The corresponding equation has real roots.

Solution 2, by Corneliu Manescu-Avram, Daniel Vacaru, and the Missouri State University Problem Solving Group (done independently).

If the discriminants of all three equations are negative, then

$$q^2 < pr, \quad p^2 < rq, \quad r^2 < qp,$$

whence

$$p^2q^2r^2 < (pr)(rq)(qp) = p^2q^2r^2,$$

which is a contradiction. Hence at least one equation must have real roots.